Topology in physics 2019, exercises for lecture 12

- The hand-in exercises are exercises 2 and 4. However, as we did not cover the material for exercise 4 in the lecture, you may choose to hand in that exercise as part of the exercises two weeks from now (deadline Wednesday May 22). The grade for week 12 will be the average of the grades for those exercises either way. If you take the 6EC version of this course, you still should hand in exercise 4, either this week or two weeks from now.
- Please hand in electronically at `topologyinphysics2019@gmail.com` (1 pdf, readable!)
- Deadline is Wednesday May 8, 23.59.
- Please make sure your name and the week number are present in the file name.

Exercises

Exercise 1: More on the heat kernel

Let us consider the heat kernel for an elliptic operator $\Delta : \mathcal{H} \to \mathcal{H}$, where $\mathcal{H}$ is a Hilbert space (say over $\mathbb{R}^n$) of quantum states. We can now express the heat kernel in coordinate basis as

$$h(x, y, t) = \langle x | e^{-t\Delta} | y \rangle \,, \tag{1}$$

where $|x\rangle$ and $|y\rangle$ are position eigenstates.

a. Show that this function satisfies the heat equation

$$\left(\frac{\partial}{\partial t} + \Delta_x \right) h(x, y, t) = 0, \tag{2}$$

where Δ_x indicates that we act with Δ on the x-variable.

The reason for the name "heat kernel" is that we can use the above function as a "kernel" to generate *any* solution to the heat equation. It is a standard result in the theory of differential equations that such a solution $u(x, t)$ is completely determined by its initial conditions $u(x, 0) \equiv u_B(x)$.

b. Show that the function

$$u(x, y) = \int_{\mathbb{R}^n} h(x, y, t) u_B(y) dy \tag{3}$$

satisfies the heat equation and has the initial conditions $u_B(x)$.

$\star$ Exercise 2: An alternative way to compute the index

We have seen in the lectures that the index of an operator D can be computed using the operator $e^{-t\Delta}$. Here, we investigate an alternative but similar computation. Assume we have an elliptic, Fredholm operator $D : \Gamma(M, E) \to \Gamma(M, F)$ and intruduce its adjoint operator $D^\dagger$ and the Laplacians $\Delta_E = D^\dagger D$ and $\Delta_F = DD^\dagger$. Moreover, we introduce the function

$$I_E(s) = \mathrm{Tr}\,\left(\frac{s}{\Delta_E + s}\right) \tag{4}$$

and similarly define $I_F(s)$.

a. Show that for $s > 0$, $I_E(s) - I_F(s)$ is independent of s.

b. Show that $I_E(s) - I_F(s) = \mathrm{ind}(D)$.

You may assume without proof in this exercise that all traces involved exist and are finite numbers. This does require the condition that $s > 0$, though! (Why?)

Exercise 3: The fermionic harmonic oscillator

Introduce operators c and $c^\dagger$ that have the anticommutation relation

$$\{c, c^\dagger\} = 1. \tag{5}$$

Define $|0\rangle$ by $c|0\rangle = 0$ and $|1\rangle = c^\dagger|0\rangle$. Note that we do *not* impose that $\{c, c\} = \{c^\dagger, c^\dagger\} = 0$ yet.

a. Introduce a state $|2\rangle$ and compute its norm.

b. Argue from (a) that it is natural to impose $\{c, c\} = \{c^\dagger, c^\dagger\} = 0$. What happens if we don't?

As in the lecture, we now introduce the coherent state $|\theta\rangle = |0\rangle + \theta|1\rangle$.

c. Show that the completeness relation can now be written as

$$I = \int d\theta^* d\theta\ |\theta\rangle\langle\theta| e^{-\theta^*\theta} \tag{6}$$

d. Similarly, show that the trace of $e^{-\beta H}$ can be written as

$$\mathrm{Tr}\ e^{-\beta H} = \int d\theta^* d\theta\ e^{-\theta^*\theta}\langle -\theta| e^{-\beta H}|\theta\rangle. \tag{7}$$

That is, we need *anti-periodic* boundary conditions to define the trace as a Grassmann integral.

⋆ Exercise 4: A supersymmetric Lagrangian

As in the lecture, we study the Lagrangian

$$L = \frac{1}{2}\dot{x}^i \dot{x}_i + \frac{i}{2}\psi^i \dot{\psi}_i \tag{8}$$

a. Show that the transformation $\delta x^i = i\epsilon\psi^i$, $\delta\psi^i = -\epsilon\dot{x}^i$ changes the Lagrangian by a total t-derivative.

b. Using the canonical commutation relations between coordinates and momenta, show that the transformation in (a) is generated by the operator $Q = \psi^i \dot{x}_i$.